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way with too buoyant disciples. One ought now to compare Peirce's statement with a curious note of Dewey's—another case where a very significant statement is relegated to a foot-note: "The point of view which is here presented is, of course, distinctly pragmatic. I am not quite sure, however, of the implications of certain forms of pragmatism. They sometimes seem to imply that a rational or logical statement is all right up to a certain point, but has fixed external limits, so that at critical points recourse must be had to considerations which are distinctly of an irrational or extra-logical order. . . . It is just the opposite which I am endeavoring to sustain, *viz.* . . . " (p. 10). Yes, of course. There is the dilemma precisely: James sees well enough—without heralding it too loud—that in following logic to the end, there is no pragmatism left, and in order to be allowed to remain a pragmatist, it is necessary, at one certain point—to jump.[8] While Dewey, who obstinately persists in remaining true to logic, keeps of pragmatism nothing but the word, and lands in the *science des mœurs.*

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INFINITY AND THE GENERALIZATION OF THE CONCEPT OF NUMBER

IT is not very often that the student of philosophy has occasion to wander afar in the country of the mathematician, or that the mathematician is moved to follow the paths pioneered by the philosopher. Sometimes, however, the lack of definite boundaries suggests to each a more careful investigation of the other's habitat; for example, when the notion of infinity is under consideration. Here we should have not a trysting-place, but a battlefield, were it not for the fact that the underbrush hides the scouts from each other's sight, making it possible for the philosophers to report a treacherous and difficult pass, where the mathematicians, examining the same country, behold a clear and unobstructed roadway. Occasionally a straggler falls in with the other party and returns to oppose his report to that of his fellows, as gotten from new vantage-ground. This Professor Royce has done in taking over the mathematical concept of infinity to philosophy, and Mr. Bertrand Russell, in formulating paradoxes arising from the infinite with philosophical dialectic. To the present writer, who has frequently been a sojourner with the mathematicians, it seems to be an important problem to determine what the facts really are, and, as preliminary to the task, he sets himself the problem of studying the mathematicians' report and examining the use that is, and that can be, legitimately made of it. In the question of interpretation, mathematics is especially open to error, for, from the nature of the case, its methods are preeminently suited to generalization, but are weak when it is a question of discovering subtleties of implication.

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[8] "Materialism and agnosticism [read simply 'determinism'] even were they true, could never gain universal and popular acceptance, for they both, alike, give a solution of things which is irrational to the practical third of our nature" ("Will to Believe," p. 126). *Even were they true:* a man who would really believe in the possibility of proving logically that they are *not true,* would not have recourse to such hypothetical argumentation.

In all mathematical writings two sorts of infinities are distinguished, the cardinal and the ordinal infinite. This latter, however, need not concern us, as it aims directly at the definition of a certain kind of order in which the elements of certain classes may be put, while we are interested in the nature of the class or system itself, without reference to the order of its elements. The definition of the cardinal infinite is supposed to give us what we require.

To quote at once the classic definition: "A system S is called 'infinite' when it is similar (*i. e.*, can be put in one-one correspondence) with a proper part of itself; in the contrary case S is called a 'finite' system."[1] At first sight it seems as if nothing could be clearer than this definition, and it is generally assumed to be irreproachable. Unfortunately, in its applications it is not always so, and this will soon appear.

One great virtue of modern mathematics is that it consciously aims at avoiding self-contradictions and recognizes that the only possible proof of consistency lies in showing something existent to which its definitions and postulates apply, *i. e.*, something real, or at least imaginable, with respect to which we can interpret the definitions, or do all the things indicated by the postulates. This criterion is easily applied when geometry is in question, and it must also be applied to the definition of infinity. Consequently, we find with Dedekind: "Theorem: Infinite systems exist. Proof: My own realm of thoughts (*Gedankenwelt*); *i. e.*, the totality S of all things that can be objects of my thought, is infinite. For if s is an element of S, it follows that the thought s, *viz.*, the thought that S can be an object of my thought, is itself an element of S." Professor Royce[2] suggests the case of a perfect map of England drawn on the surface of England as a demonstration of this sort. The class of whole numbers and of the points on a line has also been used. It is admitted that all these cases make appeal to a principle of complete induction, but this principle is probably grounded in logical intuition and its validity can hardly be called in question. Accordingly, mathematicians, and sympathizing philosophers, are satisfied that the existence of an actually infinite class has been shown.

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[1] Dedekind, "Was sind und was sollen die Zahlen," p. 17.

[2] "The World and the Individual," Vol. I., p. 503.

But in what sense? The thought collection is never realized, the maps are never completed, the numbers and points are never put into one-one correspondence with a part of themselves. Nor do we mean to imply any quibble concerning time. From a practical point of view a finite class may be easily indicated such that a temporal correlating of it with some other finite class could not be actually carried to an end. But it would be absurd to introduce such considerations here. The important thing is that, logically, an essentially uncompleted process is implied in the definition of an infinite class, for if there is any point at which a next and, as yet, untaken step is not defined the collection would thereby become not infinite, but finite. All that we have, then, is a certain type of open process, and certain classes admitting of a correlating process of this type. We have not even advanced far beyond Locke's conception when he says: "For, as our idea of infinity being as I think, *an endless growing idea,* but the idea of any quantity the mind has, being at that time *terminated* in that idea, . . . to join infinity to it, is to adjust a standing measure to a growing bulk; and therefore I think it is not an insignificant subtlety, if I say, that we are careful to distinguish between the idea of the infinity of space and the idea of a space infinite." In exactly Locke's sense, the only infinity defined as consistent, and therefore existing for the mathematician, is the "infinity of a system" and not "a system infinite." Also the characteristic of such classes is that they are not "static," but "dynamic," that is, in them new elements are always bubbling up. The infinity of a class demands a genuine growth in Locke's sense, for this is the significance of the open correlating process whereby infinity is defined. Satisfaction of the definition implies merely that the class possesses what might be termed an unlimited power of spontaneous self-generation.

Had the mathematician been willing to take his definition in this sense, there would have been nothing of which we could complain. To say that a class or system of things submits to a certain correlating process of subcollections of its elements with the whole, is to posit a property of the class, and this property may be taken as the essence of the class of classes that possess it, and this class contrasted with the class of classes that do not possess it. It is to be noted, however, that this opposed class, the finite, is only defined negatively; and there is danger when we begin to transfer properties from one to another in a pair of logically opposite classes, for a property may have a connotative accompaniment, essential in the one, but not transferable to the other. This is what the mathematicians have neglected to consider. We shall find that, because the cardinal number is defined through a correlating process in the field of the finite, and because correlating processes also have meaning in the region of the infinite, or transfinite, they have carried over the definition of the cardinal number, without considering that it might make a difference that the correlation, on the one side, implied a closed process and the presence of all the elements, and, on the other side, an open process and unlimited pregnancy with new elements.

Cantor[3] has expressed the belief that "mathematics is completely free in its development, and has only to pay attention to the self-evident condition that its conceptions are both free from self-contradiction and, in determinate relation, fixed by definitions, to the conceptions already present and verified. In particular, with the introduction of new numbers, it has only to give definitions of them by which such a definiteness, and, under circumstances, such a relation to the older numbers, are afforded that they can be distinguished from one another in given cases." There are many qualifying phrases here, but it is interesting to see how the task is carried out. Let us examine, in the case of cardinal numbers, the operations of making a transition from the finite to the infinite.

First, something about the finite cardinals. Two classes are said to be equivalent when they can be brought in one-one correspondence, and a class of such equivalent classes is said to be their cardinal number. That is, cardinal number connotes a way of establishing a one-one correspondence between classes of a particular sort, which amounts, practically, to the common-sense notion that it expresses the result of counting them. But with the finite numbers, the connotation is also involved that the elements are all there, and the process complete. Therefore, of two different cardinal numbers, it will happen that one can be called less than the other, when its process is insufficient to correlate the elements of the class of which it is the cardinal number with the elements of another class than which it is said to be less. In a similar fashion, addition appears by the superposition of one process on another which has been completed, and the definition of the result as the outcome of a new process, such that those classes of which it is the cardinal number may be called equal to the sum of the original classes.

For the finite numbers this is simple enough, and to Cantor's mind the extension to the transfinite involves no difficulty. Just as the finite cardinal number gives the type of certain classes, so there can be a type of those classes such that their elements can be put in one-one correspondence with a part of themselves, and we have a cardinal number of an infinite class, analogous to the definition of the cardinal number of a finite class. It is unquestionable that there exist such classes of similar classes, and that therefore this cardinal number has some meaning. Furthermore, there appear to be infinite classes which do not have the same cardinal number, for whereas the collection of finite numbers and the collection of all the points in space can each be put in one-one correspondence with a part of itself, no one has yet succeeded in putting them in one-one correspondence with each other, and Cantor has proofs, which have not been refuted, that it can not be done. Granting, then, that different transfinite cardinal numbers, in some sense, exist, a very interesting situation is found, for there is no proof that these different numbers can be ordered in a linear series, *i. e.*, that they are analogous to the finite numbers in being distinguishable for the purpose of counting. To be sure, Cantor can prove that when any class is given, a class can be constructed that shall have a greater cardinal number than the given class. That might mean, however, that the two "growing" processes are so related that one must always be in advance of the other. An illustration of this theorem by Professor Huntington[4] points quite definitely to this interpretation. "For example, let C denote the class of elements in a linear continuum, say, the class of points on a line one inch long; and let C' denote the class of all possible 'bi-colored rods' which can be constructed by painting each point of the given line either red or blue. Then the class of rods C' has a higher cardinal number than the class of points C, as may be proved as follows: In the first place, *C is equivalent to a part of C';* for example, to the class of rods in which one point is painted red and all the other points blue. Secondly, *C is not equivalent to the whole of C';* for, if any alleged one-one correspondence between the rods and the points were proposed, we could at once define a rod which would not be included in the scheme: namely, the rod in which the color of each point x is opposite to the color of the point x in the rod which is assigned to the point x of the given line; this rod would differ from each rod of the proposed scheme in the color of at least one point." In other words, one scheme of "growth" is imposed on another in such a fashion that the combined result is always in advance of that logically given with the first, and the misinterpretation of this as meaning that the second collection is greater than the first arises from carrying over to the transfinite a connotation true only of the finite, namely, that the elements of the class are given statically, a thing which our examination of the definition of infinity showed us is logically impossible. "To be given" has an ambiguity in meaning

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[3] "Grundlagen einer allgemeinen Mannigfaltigkeitslehre," Leipzig, 1883, p. 33.

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[4] Huntington, "Annals of Mathematics," 2d series, Vol. 7, 1905, p. 39.

ing, and the new concept is not "in determinate relations, fixed by definitions, to the conceptions already present and verified." Have we, then, a right to call these new types numbers?

The illusion that we have this right is strengthened by the fact that Cantor develops his theory in close connection with the concept of series, although it is understood that the cardinal number here is the "cardinal number of the *class of elements which occur in the series,* without regard to their order."[5] Different types of series are definable, and it is possible to order some of these types of series.[6] Also, there will be a cardinal number of the elements of these ordered series of series. But the very fact that the transfinite numbers are dependent for definition on series of series, while vouching for their possibility in some sense, also points to the explanation of them given above as due to the superimposition of process on process.

It is hard for the mathematician to see it this way. To him, the objection is insuperable that all the elements of a transfinite collection are given when the collection is defined, and all the points of a line are given when the line is given. To deny this seems to him absurd. Yet the denial is possible when we get rid of the ambiguity of the word *given.* Suppose a line, and there is a possibility of finding as many points as you please. The line is infinitely pregnant with points, it is analyzable into points, but *it can not be synthetically constructed of points.* If it were, the points would be given in the assumed sense of *given,* which, with Locke, we believe to be invalid. The linear continuum can be mathematically defined, and it gives us such a method of analyzing a line into points that no other process of analyzing it shall be continually in advance of this one, and so seem to give us more points. We have, then, practically reduced the line to points, but it is the error specified by Locke to consider these points as statically given.

On the basis of these considerations, it is evident that the most that can be meant by Cantor's transfinite numbers is that, given an open process, the type of infinity, it is possible to define a modification of this process through elements related to the elements originally implied in such fashion that the combined implication of these processes shall be essentially non-equivalent to the implication of the original process. But this does not guarantee that there are more elements in the one case than in the other, any more than the fact that to correlate a set of things by painting them both red and green, which involves the putting on of one color before the other and so double work, means that there are more things than there would have been had they been merely painted green, or than the fact that the

[5] Huntington, *loc. cit.*, p. 40.

[6] *Cf.* Huntington, *loc. cit.*, p. 35

kitten fails to catch its tail means that the tail goes faster than the cat.

And this brings us back to our starting place. A cardinal number is not merely the name of a correlation process, but it is the name of a closed correlation process applied to elements that are logically possible together. In giving up these two essential connotations the transfinite loses a part of the essence of number and has no claim to the name unless we wish to use the name *cardinal number* for different sorts of arrangements of elements, or, in other words, for qualitatively different collections.

If the analysis of this paper is correct, mathematicians have committed the fallacy of converse accident in their generalization of the concept of number, and only the finite numbers are legitimate. Such success as has been attained must, therefore, be explained through the correctness of a limited aspect of the theory of the transfinite, *i. e.*, the study of the relations of ordinal types. The difficulties of the theory, a most marvelous budget of paradoxes, vanish under such limitations. In fact, some such limitation has already been suggested for this very reason, although explanation of it, other than the need of avoiding contradiction, has not been given.[7] However, the scope of this paper will not permit us to touch upon the paradoxes of the transfinite.

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DISCUSSION

ULTIMATE HYPOTHESES IN PSYCHOLOGY

IN his very interesting discussion of "ultimate hypotheses in psychology," published in this JOURNAL August 13, Professor Tawney seems to me to overstate the divergence of my view from his. His teaching (in an earlier paper,[1] which, however, he here cites with approval) that self-awareness is a constituent of all consciousness is, I think, the essence of my contention that the self is basal fact of all consciousness (and not merely of consciousness at its higher levels). In the paper just published in this JOURNAL Dr. Tawney makes this "vague self-awareness . . . identical with the immediate experience of value." I question the expediency of the identification, but this may be a difference of terminology only.

In comment upon Dr. Tawney's criticisms of my conception, I have two points to urge. In the first place, I wish that he would

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[7] *Cf.* B. Russell, *Proceedings of the London Mathematical Society*, Series 2, Vol. IV., Part 1, pp. 29-53.

[1] "Feeling and Self-awareness," *Psychological Review*, Vol. IX., pp. 570 ff.